A New Mechanism for Efficient and Fair Course Allocation: Data, Theory, Computation, Experiments, and Practical Implementation

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Other examples: assigning interchangeable workers to tasks or shifts; leads to salespeople; takeoff and landing slots to airlines; shared scientific resources amongst scientists; players to teams

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- Divisible Goods → Classic Fair Division Problem
  - Theory: Steinhaus 1948 ...

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Dictatorship Theorem. The only mechanisms that are ex-post Pareto efficient and strategy-proof are dictatorships (Klaus and Miyagawa, 2001; Papai, 2001; Ehlers and Klaus, 2003; Hatfield, 2009)

- What is a dictatorship in this context?
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- Other negative results for closely related problems (Sonmez, 1999; Konishi, Quint and Wako, 2001; Klaus and Miyagawa, 2001; Manea, 2007; Kojima, 2012)
- Impossibility theorems are even more severe if we seek ex-ante Pareto efficiency (Zhou, 1990)

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- Papai (2001, p. 270): "[t]he implications are clear (...) if strategic manipulation is an issue, one should seriously consider using a serial dictatorship, however restrictive it may seem."
- Ehlers and Klaus (2003, p. 266): "[a] practical advantage of dictatorships is that they are simple and can be implemented easily. Furthermore, they are efficient, strategyproof (...). They can be considered to be 'fair' if the ordering of the agents is fairly determined; for instance by queuing, seniority, or randomization."
- Hatfield (2009, p. 514): "[the] results have shown that the only acceptable mechanisms for allocation problems of this sort is a sequential dictatorship, even when we restrict preferences to be responsive (...). Although unfortunate, it seems that in many of these applications, the best procedure (...) may well be a random serial dictatorship."

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- Wrong efficiency notion.
  - What we care about is not ex-post Pareto efficiency, but ex-ante welfare
  - Ex-post Pareto is very weak if not constrained by fairness considerations (e.g. agent 1 gets the whole endowment)

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My own takeaway: there is a basic tension among concerns of efficiency, fairness, and incentives. Any new market design will involve compromise of competing design objectives.

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5. ... Now implemented in practice at Wharton
   - Called "Course Match". Used as of Fall 2013.

Eric Budish and Estelle Cantillon, 2012. "The Multi-Unit Assignment Problem: Theory and Evidence from Course Allocation at Harvard". *American Economic Review*

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  2. Students are randomly ordered by the computer
  3. Students are allocated courses one at a time, based on their reported preferences and remaining availability
     - Rounds 1, 3, 5, ...: ascending priority order
     - Rounds 2, 4, 6, ...: descending priority order

The Draft: Formal Theoretical Properties

It is easy to show that the draft is not strategy-proof (cf. Example 1 of BC 2012)

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$$\text{dictatorship} > \text{draft}$$

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   - *Does the lack of strategy-proofness actually matter in practice?*

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   - *Does the lack of strategy-proofness actually matter in practice?*
2. We can simulate equilibrium play of the (Random Serial) Dictatorship, and so can compare the two mechanisms
   - *Should HBS switch to the strategy-proof alternative?*

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RSD-Truthful	8.74	49%	29.7%

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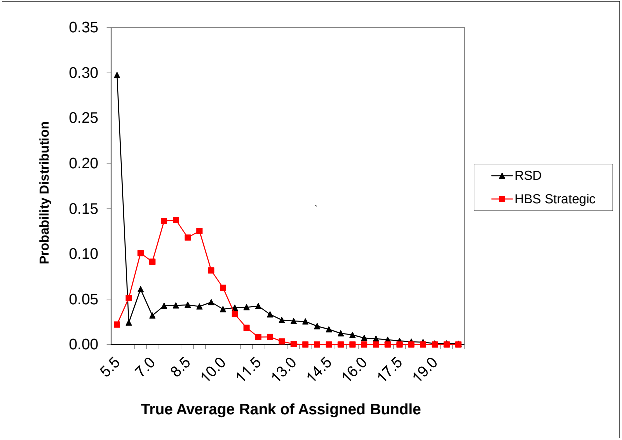

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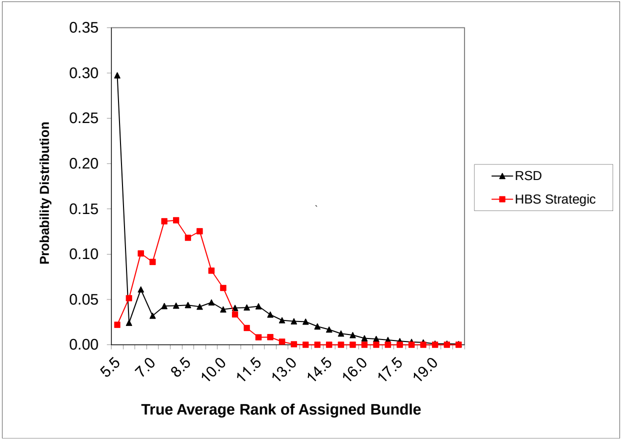

- HBS Second-Order Stochastically Dominates RSD
- Implication: social planner prefers HBS to RSD if students have average-rank preferences and are weakly risk-averse

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  - Note: truthful play is an eqm in this example

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- Important note: unattractiveness of RSD does not depend on risk preferences. Even risk-neutral agents regard a "win a little, lose a lot" lottery as unappealing.

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- So much so that the HBS lottery over *in*efficient allocations looks more attractive ex-ante than RSD.
  - No efficiency-fairness tradeoff
  - Ex-post efficiency need not even proxy for ex-ante efficiency

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- And we also find that the welfare costs of using a strategyproof dictatorship appear to be much larger than the welfare costs of manipulability
- Overall, suggests a nuanced view of the role of strategyproofness in design, and the need for second-best alternatives to exact SP (eg "strategy-proofness in the large", Azevedo and Budish, 2013)

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  - Avoid HBS draft's critical flaw – manipulability – without triggering dictatorship theorems
- Seek a mechanism that yields a relatively equal distribution of outcomes, like the draft and unlike the dictatorship
  - Avoid dictatorship's critical flaw – severe ex-post unfairness, which harms ex-ante welfare

Budish: “The Combinatorial Assignment Problem: Approximate Competitive Equilibrium from Equal Incomes” (*JPE*, 2011)

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- Based on an old idea from general-equilibrium theory called Competitive Equilibrium from Equal Incomes (Foley 1967, Varian 1974)

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- It is easy to see that existence is problematic.

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- Exact CEEI: $\alpha = \beta = 0$

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- Other extreme: dictatorships can be interpreted as exact CE ($\alpha = 0$), but from arbitrarily unequal budgets
- Market administrator can assign these close but unequal budgets to agents however she likes (e.g., uniform randomly)

Key Ideas in the Proof

- Basic difficulty: agents' demands are discontinuous with respect to price, because of indivisibilities
  - Standard Arrow-Debreu-McKenzie existence results assume that preferences are continuous
- Role of σ: $\sqrt{\sigma}$ bounds the magnitude of an individual agent's demand discontinuity
  - At worst, a small change in price can cause an agent's demand to change from one bundle of objects to an entirely disjoint bundle of objects
- Role of unequal budgets: mitigate how individual agent demand discontinuities aggregate up into aggregate demand discontinuities
  - If agents have same budgets: discontinuities occur at the same points in price space. Can have "large" discontinuities: magnitude $N\sqrt{\sigma}$
  - If agents have distinct budgets: possible to change one agent's choice set without changing *all* agents' choice sets
  - Maximum magnitude of a discontinuity in aggregate demand is $M\sqrt{\sigma}$

Key Ideas in the Proof

- Can now apply an off-the-shelf approximate fixed point theorem (Cromme and Diener, 1991) to get market clearing error of at most $M\sqrt{\sigma}$
- Hard part of the proof is reducing the worst-case bound to $\frac{\sqrt{\sigma M}}{2}$
  - Exploits geometric structure of aggregate demand in a neighborhood of any p: zonotope
  - We can make an affordable / not-affordable decision for each agent-bundle pair where affordability near p is in question
- Payoff to reducing the bound is meaningful for practice:
  - Semester at HBS: $M = 50$, $\sigma = 5$
  - $M\sqrt{\sigma} = 112$
  - $\frac{\sqrt{\sigma M}}{2} = 11$

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  - Observe how budget inequality recovers existence (in this case exactly)

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- Fair share is not well defined with indivisibilities: what is $\frac{1}{N}$ of the endowment?
- Envy freeness will be impossible to guarantee with indivisibilities. What if there is some single star professor whose course all students very badly want to take?

Previous Approaches to Outcome Fairness with Indivisibilities

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My approach

- Keep my problem as is, but weaken the criteria to accommodate indivisibilities in a realistic way

Maximin Share Guarantee

Definition. Agent i's **maximin share** is the consumption bundle

$$\max_{(u_i)}\{\min_{(u_i)}\{x_1, \ldots, x_N\}\}$$

where the $\max(\cdot)$ is taken over all feasible allocations. Any allocation in which all N agents get a bundle they prefer to their maximin share is said to satisfy the **maximin share guarantee**

- Divide-and-choose interpretation
- Rawlsian guarantee from what Moulin (1991) calls a "thin veil of ignorance"
- Coincides with fair share if goods divisible, prefs convex and monotonic

Envy Bounded by a Single Good

Definition. An allocation $\mathbf{x}$ satisfies **envy bounded by a single good** if, for any two agents i, i' , either:

1. $u_i(x_i) \geq u_i(x_{i'})$
2. $u_i(x_i) \geq u_i(x_{i'} \setminus \{j\})$ for some $j \in x_{i'}$

- In words: if student i envies student i' the envy is bounded: by removing some single good from i''s bundle we would eliminate i's envy
- Coincides with envy-freeness in a limit as consumption bundles become perfectly divisible

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  - Whichever agent goes first gets both diamonds

Diamonds and Rocks Revisited

- In diamonds and rocks example, Approximate CEEI satisfies both fairness criteria
  - Each agent's maximin share is {Small Diamond, Pretty Rock}.
  - The agent who gets {Small Diamond, Pretty Rock} envies the agent who gets {Big Diamond, Ugly Rock}, but the envy is bounded by a single good (the big diamond)
- Dictatorships satisfy neither criteria
  - Whichever agent goes first gets both diamonds
- By contrast, in single-unit assignment problems (e.g., one diamond, one rock), dictatorships actually do satisfy both criteria
  - Dictatorships are frequently used in practice for single-unit assignment problems (school choice, housing assignment)
  - The fairness properties help us to make sense of the empirical patterns of dictatorship usage. Useful external validity check.

Fairness Theorems for Approximate CEEI

Theorem 2: If $\beta < \frac{1}{N}$ then $\mathbf{x}^*$ guarantees each agent their $N+1$-maximin share (maximin share in a hypothetical economy with one additional agent).

Intuition for proof:

1. If $\beta < \frac{1}{N}$ then even poorest student has $> \frac{1}{N+1}$ of the income endowment
2. If $\mathbf{p}^*$ is an exact competitive equilibrium, then the goods endowment costs weakly less than the income endowment
3. So, if $\mathbf{p}^*$ is an exact c.e., each student can afford some bundle in *any* N+1-way split
4. Hence, each student must be able to afford some bundle weakly preferred to her N+1-maximin share

The full argument is a bit messier because $\mathbf{p}^*$ might be an approximate c.e.

Fairness Theorems for Approximate CEEI

Theorem 3: If $\beta < \frac{1}{k-1}$ then $\mathbf{x}^*$ satisfies envy bounded by a single good.
Intuition for proof:

1. Suppose i envies j . Then

$$1 \leq b_i^* < \mathbf{p}^* \cdot x_j^* \leq b_j^* \leq \frac{k}{k-1}$$

2. Since x_j^* contains at most k goods, one of them must cost at least $\frac{1}{k-1}$. i can afford the bundle formed by removing this good from x_j^*
3. Hence, i must weakly prefer her own assigned bundle to the bundle formed by removing this single good from x_j^*, so her envy is bounded.

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4. Allocate each student her demand at $\mathbf{p}^*$ based on her reported preferences

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  - Mechanisms that are not SP-L: Boston mechanism, priority-match algorithm, pay-as-bid auctions, HBS draft

Properties of the Approximate CEEI Mechanism

Efficiency
Ex-post efficient, but for small error

Fairness
N+1 Maximin Share Guarantee
Envy Bounded by a Single Good

Incentives
Strategy-proof in the Large

TABLE 1
COMPARISON OF ALTERNATIVE MECHANISMS

Mechanism	Efficiency (Truthful Play)	Fairness (Truthful Play)	Incentives	Preference Language
A-CEEI mechanism	Pareto efficient with respect to allocated goods Worst-case allocation error is small for practice and goes to zero in the limit	$N+1$—maximin share guaranteed Envy bounded by a single good	Strategyproof in the large	Ordinal over schedules
Mechanisms from practice:				
Bidding points auction (Sonmez and Ünver 2003)	If preferences are additive separable, Pareto efficient but for quota issues	Worst case: get zero goods	Manipulable in the large	Cardinal over items
HBS draft mechanism (Budish and Cantillon, forthcoming)	If preferences are responsive, Pareto efficient with respect to the reported information	If preferences are responsive and $k = 2$, maximin share guaranteed If preferences are responsive, envy bounded by a single good	Manipulable in the large	Ordinal over items
Univ. Chicago primal-dual linear program mechanism (Graves, Schrage, and Sankaran 1993)	Pareto efficient when preference-reporting limits do not bind	Worst case: get zero goods	Manipulable in the large	Cardinal over a limited number of schedules
Mechanisms from prior theory:				
Adjusted winner (Brams and Taylor 1996)	If preferences are additive-separable, Pareto efficient	Worst case: get zero goods	Manipulable in the large	Cardinal over items

Descending demand procedure (Herreiner and Puppe 2002)	Pareto efficient	Does not satisfy maximin share guarantee or envy bounded by a single good	Manipulable in the large	Ordinal over schedules
Gale-Shapley enhancement to the BPA (Sönmez and Ünver 2003)	If preferences are additive separable, Pareto efficient	Worst case: get zero goods	Bidding phase: manipulable in the large Allocation phase: strategyproof in the large	Bidding phase: cardinal over items Allocation phase: ordinal over items
Geometric prices mechanism (Pratt 2007)	If von Neumann–Morgenstern preferences are additive separable, Pareto efficient	Worst case: get zero goods	If von Neumann–Morgenstern preferences are additive separable, strategyproof in the large	Cardinal over items
Minimize envy algorithm (Lipton et al. 2004)	Algorithm ignores efficiency	If preferences are additive separable, envy bounded by a single good	Manipulable in the large	Cardinal over schedules
Serial/sequential dictatorship (cf. Pápai 2001)	Pareto efficient	Worst case: get k worst goods	Strategyproof	Ordinal over schedules
Other solution concepts:				
Maximin utility	Pareto efficient	Worst case: get approximately zero goods (if a hedonist and all other agents are depressives)	Manipulable in the large	Cardinal over schedules
Utilitarian solution	Pareto efficient	Worst case: get zero goods (if a depressive and all other agents are hedonists)	Manipulable in the large	Cardinal over schedules

Othman, Sandholm and Budish: "Finding Approximate Competitive Equilibria: Efficient and Fair Course Allocation" (AAMAS, 2010)

Computational Procedure for Approximate CEEI

- Theorem 1 of Budish (2011) shows Approx CEEI prices *exist*
- Does not show how to find them
  - Proof is non-constructive
  - Approximate Kakutani fixed point

Computational Procedure for Approximate CEEI

- Othman, Sandholm and Budish (2010) develops a computational procedure for finding approximate market-clearing prices.
- Two level procedure:

1. Agent Level: computing students' demands at a candidate price vector $\mathbf{p}$
   - NP Hard: knapsack problem
   - But: doable in practical sized problems, and highly parallelizable (Hayek)
2. Price Level: searching through price space for approximate market-clearing prices
   - We use a method called "Tabu Search"
   - Departure point is the Tatonnement process: $\mathbf{p}^{t+1} = \mathbf{p}^t + \mathbf{z}(\mathbf{p}^t)$
   - Note: no guarantees ... open question whether there is a better method

Performance of Approximate CEEI on the HBS Data

1. Ex-Post Efficiency: Market-clearing error is small
   - Mean error is plus/minus 1 seat in 6 courses (out of ~50 courses / ~5000 course-seats per semester)
   - Implication: outcomes are nearly ex-post efficient

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   - Implication: a utilitarian social planner should prefer Approximate CEEI to either of these alternatives
   - That is, even a planner who does not care about fairness per se, only welfare, prefers Approximate CEEI