The Combinatorial Assignment Problem

(1) Theory and Evidence from Course Allocation at Harvard (w Estelle Cantillon)

(2) Approximate Competitive Equilibrium from Equal Incomes

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Columbia University, November 10 2009

The Combinatorial Assignment Problem

General question: How can we divide a set of indivisible objects amongst a set of agents without using monetary transfers, in a way that is efficient, incentive compatible, and fair?

Specific instance: Course Allocation at Universities

- The indivisible objects are seats in courses
- Each student requires a bundle of courses
- Exogenous restriction against monetary transfers (even at Chicago!)

Other examples: assigning interchangeable workers to tasks or shifts; leads to salespeople; takeoff and landing slots to airlines; shared scientific resources amongst scientists; players to teams

The Combinatorial Assignment Problem

Relation to the Literature

Combinatorial assignment is one feature removed from market design problems that have received considerable attention and have compelling solutions

- No restriction on money → Combinatorial Auction Problem (Vickrey 1961 ...)
- Single-Unit Demand → School/House Assignment Problem (Shapley and Scarf 1974 ...)
- Two-Sided Preferences → Matching Problem (Gale and Shapley 1961 ...)
- Divisible Goods → Classic Fair Division problem (Steinhaus 1948 ...)

Impossibility Theorems

The only mechanisms that are ex-post Pareto efficient and strategyproof are dictatorships (Papai 2001; Ehlers and Klaus, 2003; Hatfield 2007)

- What is a dictatorship in this context?
  - Alice chooses her favorite bundle of courses
  - Betty chooses her favorite bundle of courses, out of those not yet at capacity
  - ...
  - Zoe chooses her favorite bundle of courses, out of those not yet at capacity
- Other negative results for closely related problems (Sonmez, 1999; Konishi et al, 2001; Klaus and Miyagawa, 2001; Manea, 2007; Kojima, forth.; Che and Kojima, forth.)
- Impossibility theorems are even more severe if we seek ex-ante Pareto efficiency (Zhou, 1990)

Takeaway: there is a basic tension amongst efficiency, incentives and fairness. Any solution will involve compromise of competing design objectives.

Overview of Today's Talk

Given lack of progress in theory, a sensible starting point for design is to see what we can learn from mechanisms that are actually used in practice

- "Theory and Evidence from Course Allocation at Harvard" (w Estelle Cantillon)
- Lessons at 3 levels
  - Mechanism: HBS flawed but sensible; RSD worse
  - Problem: "where to look" for a solution
  - Field: strategyproofness; random mechanisms

Then, propose a new mechanism. Inspired by old idea in GE.

- "Approximate Competitive Equilibrium from Equal Incomes"
- Contributions
  - Specific Mechanism: A-CEEI
  - New criteria of outcome fairness for indivisible goods problems
  - New criterion of approximate incentive compatibility

Course Allocation at Harvard Business School

We study the course-allocation mechanism used at HBS.

1. Prima facie sensible mechanism: intuitive modification of dictatorship; widely used; "market endurance test"
2. Great data: students strategically submitted reports as well as their underlying true preferences, from a survey conducted by the HBS administration

Main results

1. Simple to manipulate in theory
2. Heavily manipulated in practice
3. Manipulations harm welfare, both ex-ante and ex-post. Magnitudes meaningful.
4. Yet, HBS preferable to Random Serial Dictatorship on measures of both ex-ante efficiency and ex-post fairness. RSD is "callous"

Environment

- Set of M courses with integral capacities $\mathbf{q} = (q_1, ..., q_M)$. No other goods in the economy.
- Continuum of N students.
- Each student requires k courses, has a vNM utility function u_s defined over permissible bundles of courses and an ordinal preference relation P_s defined over individual courses
  - We assume preferences are responsive: for any bundle x with $|x| < k$, if $cP_s c'$, then $u_s(x \cup c) > u_s(x \cup c')$
  - This is why problem is "multi-unit" not "combinatorial"
- Preferences are complete information

The HBS Mechanism

1. Students report rank-order lists over individual courses
2. Students are randomly ordered
3. Students are allocated 1 course per round for m rounds, based on their reported preferences and remaining availability
   - Rounds 1, 3, 5, ...: ascending priority order
   - Rounds 2, 4, 6, ...: descending priority order
4. Aftermarket, in which students can add courses not yet at capacity

Properties:

- Anonymous
- Ex-post efficient with respect to the reported preferences
- Fair distribution of choosing rights: no student's set of choices dominates any other's

$m = 2 \rightarrow$ HBS is only mechanism with these three properties

The HBS Mechanism is not Strategyproof

Let $m = 2$ and suppose there are 4 courses with capacity of $\frac{2}{3}$ seats each. (Responsive) preferences are as follows:

$$\frac{1}{3} \text{ students are } P_1 : a, b, c, d$$
$$\frac{1}{3} \text{ students are } P_2 : a, c, d, b$$
$$\frac{1}{3} \text{ students are } P_3 : b, a, c, d$$

Suppose everybody else plays truthfully. What is a P_3 type's best response?

- P_3: obtain $\{b, c\}$
- $\widehat{P}_3 : a, b, c, d$. Then obtain $\{a, b\}$, which is preferred.
- The profile $P_1, P_2, \widehat{P}_3$ is a Nash Eqm.
  - Type-1 and type-2 students get $\{a, b\}$ with probability $\frac{2}{3}$ and $\{b, c\}$ with probability $\frac{1}{3}$.
  - Type-3 students get $\{a, v\}$ with probability $\frac{2}{3}$ and $\{c, d\}$ with probability $\frac{1}{3}$.
  - Properties of equilibrium: over-reporting of popular course a. Course a fills up (stochastically) earlier than under truthful play.

The HBS Mechanism: Simple Manipulations Theorem

Fix strategy profile $\widehat{\mathbf{P}}$. A course is $\widehat{\mathbf{P}}-$popular if it runs out with strictly positive probability either during the initial allocation or the add-drop phase.

Theorem 1 (Simple Manipulations): Fix $\widehat{\mathbf{P}}_{-s}$. Form the strategy $\widehat{P}_s^{\text{simple}}$ by taking the first m courses in P_s and rearranging them so that $c\widehat{P}_s^{\text{simple}}c'$ whenever:

1. cP_sc' and both are $\widehat{\mathbf{P}}_{-s}-$popular or both are $\widehat{\mathbf{P}}_{-s}-$unpopular
2. c is $\widehat{\mathbf{P}}_{-s}-$popular and c' is $\widehat{\mathbf{P}}_{-s}$-unpopular

The strategy $\widehat{P}_s^{\text{simple}}$ generates weakly greater utility than truthful play P_s.

The HBS Mechanism: Equilibrium Characterization

Lemma 2 (Necessary condition for a Best-Response): Consider any NE $\widehat{\mathbf{P}}$. Suppose c is $\widehat{\mathbf{P}}-$popular. Consider student s for whom $r_s(c) \leq m$. Then one of the three following conditions must hold:

(i) c is placed before all $\widehat{\mathbf{P}}-$unpopular courses in $\widehat{P}_s$

(ii) student s gets c for sure with $\widehat{P}_s$

(iii) he gets c with probability zero and moving it up to the position of the first unpopular course on $\widehat{P}_s$ would not help.

Theorems 3, 4: Partial characterization of equilibrium in terms of what courses reach capacity, and when. Weaker than intuition from Example 1. Difficulties:

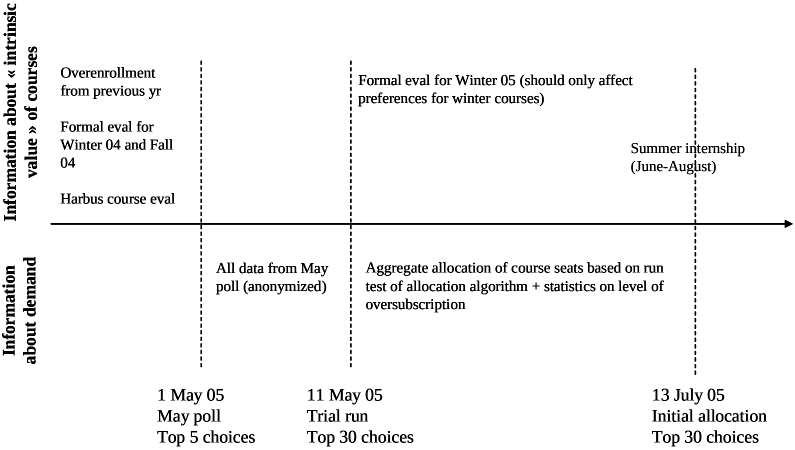
- multiple equilibria (Example 2)
- best responses depend on utility over distribution of bundles whereas input to mechanism is a ROL
- stochastic priority order

The HBS Mechanism: Efficiency and Welfare

- Strategic behavior has ex-post and ex-ante redistributive consequences
  - Students who value popular courses less benefit from the opportunity of ranking them higher
  - Students who value popular courses highly are hurt by congestion
- The HBS mechanism may be ex-post inefficient due to risk-taking by students (Example 3)
- The HBS mechanism is ex-post efficient (possible) under special circumstances (Theorem 5)
  - If truthul play is an equilibrium (Thm 2)
  - If students have lexicographic preferences (b/c they don't take risks)

The HBS Mechanism in Practice

Timing and Information



Our data

- Top 5 choices from May 1st poll for 456 students (with student identifiers)
- Submitted preferences in May 11 trial run and July 13 actual run for entire population (916) with student identifiers
- Top 30 courses from January 2006 poll for 163 students with student identifiers
- All course characteristics and information communicated to students

Evidence of strategic behavior

- Joint hypothesis: May poll data are representative of truthful preferences and July submitted ROLs are representative of "equilibrium play".
- This hypothesis is natural given the context
  - May poll: students asked by administration to tell the truth, no compelling reason not to
  - July ROL: high stakes, sophisticated players, some learning
- To support this hypothesis (beyond context), we show that the May poll preferences and the July preferences differ in a systematic way, and that this difference can be attributed to strategic behavior

Evidence of strategic behavior, cont.

- Four factors could drive differences in submissions over time

1. Idiosyncratic preference changes: should not affect aggregate distribution of ranks
2. New information about courses: correlated and persistent shock to preferences, unrelated to popularity of course
3. Social learning: correlated and persistent shock to preferences, likely to be related to popularity of courses
4. Strategic behavior: correlated but temporary shock to preferences, related to popularity of courses

Evidence of strategic behavior

- Statistical test: Extension of Mann-Whitney-Wilcoxon rank test due to Gehan (1965). Non parametric test for equality of distributions of discrete and censored data.
- Test carried out at individual course level

5% significance	N	July dd lower	No diff	July dd higher
High demand courses	20	1	**12**	**7**
Medium demand courses	37	17	20	0
Low demand courses	25	**17**	**8**	0

- Results similar if we compare May Poll versus May Trial (just 10 days apart)
- So far, evidence is consistent with either social learning or strategic behavior mainly driving the difference

Evidence of strategic behavior

Validation: Carried out same test for comparison between May poll and January poll (focus on second semester courses). Null hypothesis rejected in 29.5% of the courses (to be compared with 51%). Moreover, no systematic patterns of rejection.

	N	Jan dd higher	No diff	Jan dd lower
Low demand courses	13	1	11	1
Medium demand courses	23	5	16	2
High demand courses	8	1	4	3

- We conclude that difference between May poll preferences and July preferences are driven by strategic behavior.
- Further check: do submitted preferences in July satisfy necessary conditions for a BR? Yes, for 94% of popular-course requests

Welfare Consequences of Strategic Play

- Counterfactual exercise: compare actual outcome of HBS with outcome if students were truthful
- For counterfactual, need to construct truthful preference lists that are longer than 5 courses.
- We assume the top 5 truthful courses correspond to top 5 courses in May poll. Other courses are moved down to position 6 and below in a way that preserves relative ordering of courses not in May poll ROL
- Example:
  - May poll ROL: A, B, C, D, E
  - July run ROL: D, C, F, A, B, G, H
  - Constructed truthful preferences: A, B, C, D, E, F, G, H
- Underestimate strategic behavior: assumes non-top-5 are ranked truthfully
- Overestimate strategic behavior: interpret unobservable preference changes between May and July as strategic behavior

Welfare Consequences of Strategic Play: Ex-Post

- Caveat: incompleteness of data means we can only detect a fraction of profitable trades
- Subject to this caveat, it is wlog to focus on single course trades
- We solve a binary integer program that maximizes the number of trades

Ex-Post Pareto Improving Trades

	Mean	Std. Dev.
# of Executed Trades per Student	**1.54**	(0.04)
% of Allocated Course Seats Traded	**15.4%**	(0.31%)
% of Students Executing		
0 Trades	16.4%	(1.1%)
1 Trade	35.4%	(1.7)
2 Trades	30.5%	(1.6)
3+ Trades	17.8%	(1.3)

Welfare Consequences: Ex-Ante, Individual Level

- Challenge: partial data
- For responsive preferences, a *sufficient* condition for s to prefer truthful play to strategic play is if his distribution over bundles from truthful first-order stochastically dominates that from strategic, based on the responsiveness partial order
- How do we check this? Novel method based on bipartite matching ideas. Findings:
  - 45% prefer HBS truthful
  - 5.5% prefer HBS strategic
  - 1% indifferent
- We progressively add restrictions on preferences to get tighter comparison results. Patterns similar.
- Intuition: Asymmetry between costs and benefits of strategic play
  - Cost: congestion, harder to get favorite courses
  - Benefit: opportunism, easier to get lower-ranked courses

Welfare Consequences: Ex-Ante, Societal Level

- No Pareto comparison: some benefit, some hurt.
- Take utilitarian approach, using different assumptions on preferences.

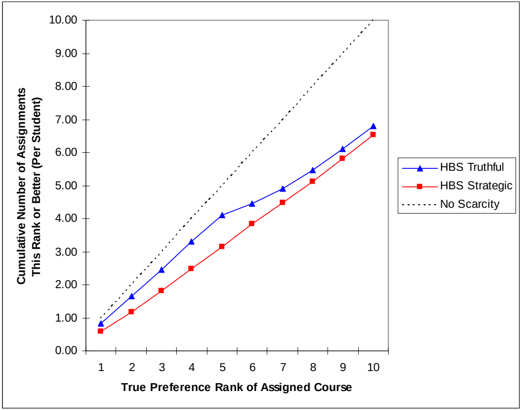
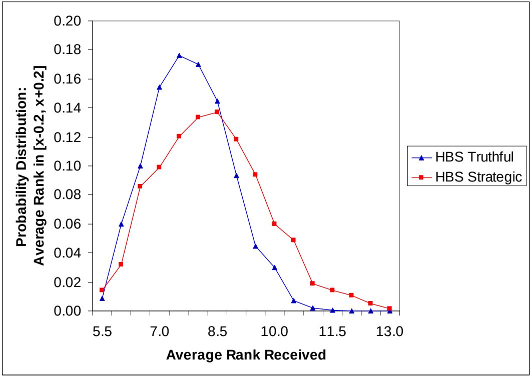


Figure:

- Implication: social planner prefers truthful play if students are

Welfare Consequences: Ex-Ante, Societal Level

- Societal distribution of Average Ranks (a proxy for utility)

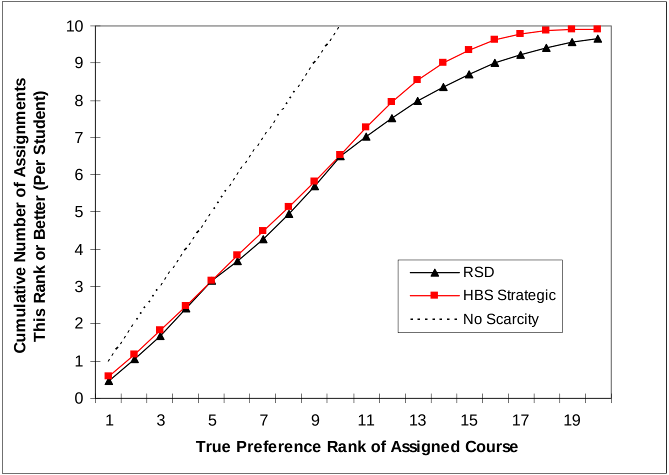


- Implication: social planner prefers truthful play if students have average-rank preferences and are weakly risk-averse. (SOSD)

HBS vs. RSD: Ex-Post versus Ex-Ante

- Since strategic behavior in the HBS mechanism harms welfare, it is natural to consider a strategyproof alternative
- Same exercise, but comparing HBS average rank distribution to that from an *equilibrium* counterfactual in which the mechanism is RSD and students are truthful
- Ex-Post: RSD is Pareto efficient, HBS is not
- Ex-Ante, Individual Level
    - Responsiveness alone: entirely indeterminate
        - Under RSD, often get "bliss bundle", often get a very bad outcome
    - If students care about average rank, 81% prefer HBS strategic
    - If students have lexicographic preferences, 75% prefer HBS strategic

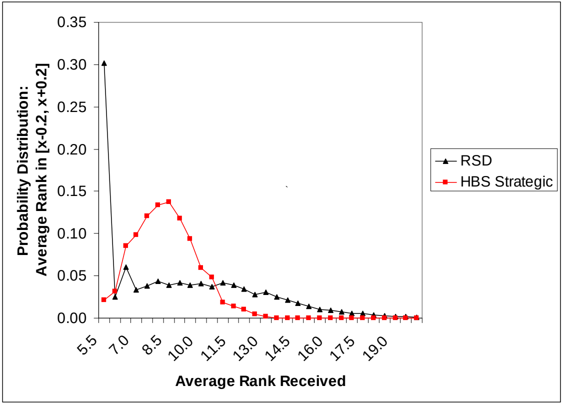
HBS vs. RSD: Ex-Ante, Societal Level



- Implication: social planner prefers truthful play if students are risk-neutral.
- This is a bit surprising. Why do risk-neutral students dislike RSD?

HBS vs. RSD: Ex-Ante, Societal Level

- Societal distribution of Average Ranks



- Implication: social planner prefers HBS to RSD if students have average-rank preferences and are weakly risk-averse. (SOSD)

HBS vs. RSD: Ex-Ante Utility

	E(Avg Rank)	% Who Get #1 Choice	% Who Get All Top 10
HBS - Truthful	7.76	83%	0.8%
HBS - Strategic	8.35	60%	1.4%
RSD - Truthful	9.84	47%	29%

Explanation: "Callousness"

- In RSD, lucky students with good random draws make their last choices independently of whether these courses would be some unlucky student's first choices
- Benefit to lucky is small; harm to unlucky is large
- Ex-post, RSD is Pareto efficient
- Ex-ante, this unavoidable callousness harms utility
  - no fairness-efficiency tradeoff
  - meaningful strategyproofness-efficiency tradeoff
- Magnitudes are large. Contrast with inefficiency of RSD in single-unit assignment (Bogomolnaia and Moulin, 2001; Pathak, 2006; Che and Kojima, 2009)

What do we Learn from the HBS Mechanism?

Lessons Learned for Market Design

1. Ex-post efficiency may not be a good proxy for ex-ante efficiency
2. Sounds a cautionary note against imposing strategyproofness as a strict design requirement

Lessons Learned for Combinatorial Assignment

1. Seek an incentives middle ground between strict strategyproofness (RSD) and simple-to-manipulate (HBS).
2. Mechanism should more resemble HBS than RSD in ex-post equality and ex-ante efficiency
   - Participants' "resources" (here, choosing times) should not be highly unequal as in RSD but rather roughly equal as in HBS
3. Fairness is central to how the HBS administration thinks about their problem. Try to articulate the notions of fairness implicit in the HBS procedure.

CAP: A-CEEI

Goal of paper: develop a "solution" to the CAP: a specific mechanism that satisfies attractive criteria of Efficiency, Fairness, and Incentives

1. Criteria of outcome fairness: *maximin-share guarantee*, and *envy bounded by a single good*
2. A specific mechanism: *Approximate Competitive Equilibrium from Equal Incomes*
    - Thm 1: Approximate CEEI is approximately ex-post efficient
    - Thm 2, 3: Approximate CEEI satisfies the proposed fairness criteria
3. Criterion of approximate IC: *Strategyproof in the Large*
4. Computational analysis of Approximate CEEI: used to assess ex-ante efficiency

This paper is something of a balancing act – both the mechanism and the criteria it satisfies are new – and I am working around impossibility theorems.

Environment

- Set of M courses with integral capacities $\mathbf{q} = (q_1, ..., q_M)$. No other goods in the economy.
- Set of N students.
- Each student s_i has a set of permissible schedules $\Psi_i \subseteq \{0,1\}^M$, and a vNM utility function $u_i : \Psi_i \to \mathbb{R}_+$

Differences versus Previous Environment

- Finite set of students
- No restrictions placed on u_i's. Comps, Subs, are allowed. No peer effects though.
- The Ψ_i's allow for time-slot constraints, curricular constraints, etc.
- The $\{0,1\}$ restriction can be relaxed but Thm 1 will be a bit weaker.

The Maximin Share Guarantee

In divisible-goods problems: an allocation satisfies the **fair-share guarantee** if each agent weakly prefers her own allocation to $\frac{1}{N}$ of the endowment. (Steinhaus, 1948)

Problem: not well defined with indivisibilities.

Definition 1. *Agent s_i's* **maximin share** *is*

$$\underline{u}_i = \max_{(x_k)_{k=1}^N} \left[\min(u_i(x_1), ..., u_i(x_N))\right]$$
$$\text{s.t. capacity constraints}$$

An allocation x satisfies the maximin-share guarantee if $u_i(x_i) \geq \underline{u}_i$ for all i.

- In words: an agent's maximin share is the maximum utility level she can guarantee herself as divider in divide-and-choose against opponents with preferences identical to her own (or adversarial opponents)
- Coincides with fair share if goods divisible, prefs convex and monotonic (Prop 2)

Envy Bounded by a Single Good

An allocation is **envy free** if each agent weakly prefers her own allocation to any other agent's allocation. (Foley, 1967)

Problem: unrealistic with indivisibilities

Definition 2. *An allocation* **x** *satisfies* **envy bounded by a single good** *if*

For any s_i, s_j*: There exists some object* $c_{j'}$ *in bundle* x_j *such that:*

$$u_i(x_i) \geq u_i(x_j \backslash \{c_{j'}\})$$

- In words: if student s_i envies s_j, the envy is bounded: by removing some single good from s_j's bundle we could eliminate s_i's envy
- Coincides with envy-freeness in a limit as consumption bundles become perfectly divisible (Prop 4)

Diamonds and Rocks Example

Example 1. Two agents. Four objects: two Diamonds (Big, Small) and two Rocks (Pretty, Ugly). At most two objects per agent.

$$\begin{aligned}\text{Maximin Share} &= \min[u(\{\text{Big Diamond, Ugly Rock}\}),\\ &\quad u(\{\text{Small Diamond, Pretty Rock}\})]\\ &= u(\{\text{Small Diamond, Pretty Rock}\})\end{aligned}$$

- The allocation in which one agent obtains {Small Diamond, Pretty Rock} and the other obtains {Big Diamond, Ugly Rock} also satisfies envy bounded by a single good
- The procedural fairness of requirement of symmetry requires randomization over who gets which bundle
- Dictatorships fail the criteria in multi-unit assignment; new impossibility results show each is incompatible with Eff + SP (Props 1, 3)

Competitive Equilibrium from Equal Incomes

Competitive Equilibrium from Equal Incomes (Foley, 1967; Varian, 1974):

1. Agents report preferences over bundles
2. Agents are given equal budgets b^* of an artificial currency
3. We find an item price vector $\mathbf{p}^*$ such that, when each agent is allocated his favorite affordable bundle, the market clears
4. We allocate each agent their demand at $\mathbf{p}^*$

It is easy to see that existence is problematic with indivisibilities. Consider the case in which agents have identical preferences.

Approximate CEEI

Definition. An allocation $\mathbf{x}^*$, budget vector $\mathbf{b}^*$ and price vector $\mathbf{p}^*$ constitute an (α, β)-**approximate competitive equilibrium from equal incomes** *(Approximate CEEI) if:*

(i) Each agent i is allocated her most-preferred bundle in her budget set $\{x \in \Psi : \mathbf{p}^* \cdot x \leq b_i^*\}$

(ii) Euclidean distance of market-clearing error at $\mathbf{p}^*$ is $\leq \alpha$
market-clearing error$_j$ $=$ demand$_j$ - supply$_j$ if $p_j > 0$
market-clearing error$_j$ $=$ max(demand$_j$ - supply$_j, 0)$ if $p_j = 0$

(iii) The ratio of the max to the min budget in $\mathbf{b}^*$ is $\leq 1 + \beta$

Exact CEEI: $\alpha = \beta = 0$

Theorem 1

Existence of Approximate CE from Approximate EI

Let k be the maximum number of courses in any permissible schedule. Define $\sigma = \min(2k, M)$ (M is the number of courses)

Theorem 1.
For any $\beta > 0$, *there exists a* $(\frac{\sqrt{\sigma M}}{2}, \beta)-$*Approximate CEEI*

Approximate Efficiency: $\frac{\sqrt{\sigma M}}{2}$ is small in two senses

- $\frac{\sqrt{\sigma M}}{2}$ does not grow with N (number of students) or $\mathbf{q}$ (number of copies of each good). As $N, \mathbf{q} \to \infty$, we get exact market clearing (error goes to zero as a fraction of the endowment)
- $\frac{\sqrt{\sigma M}}{2}$ is a small number in practical problems (especially for a worst case bound)
  - In a semester at HBS, $k = 5$ and $M = 50$, and so $\frac{\sqrt{\sigma M}}{2} \approx 11$
  - Contrast with 4500 course seats allocated per semester

N.B. a first welfare theorem indicates that an approximate CEEI allocation is Pareto efficient with respect to the set of goods actually allocated. Market-clearing error is of course inefficient.

Proof of Theorem 1: Overview

Consider a tâtonnement price-adjustment function of the form

$$f(\mathbf{p}) = \mathbf{p} + z(\mathbf{p})$$

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4. Bound market-clearing error, using the structure of demands near to $\mathbf{p}^*$. Use an exact fixed point of $F(\cdot)$ to find an approximate fixed point of $f(\cdot)$
   - Key idea: structure of demands near $\mathbf{p}^*$ has an attractive geometric structure: "zonotope" (a fancy parallelogram)

Theorem 2: Approximate CEEI Guarantees Approximate Maximin Shares

Theorem 2: if $\beta < \frac{1}{N}$ then $\mathbf{x}^*$ guarantees each agent their $N+1$-maximin share (maximin share in a hypothetical economy with one additional agent)

Intuition for proof

1. If $\beta < \frac{1}{N} \Rightarrow$ even poorest student has $> \frac{1}{N+1}$ of the income endowment
2. If $\mathbf{p}^*$ is an exact c.e. $\Rightarrow$ goods endowment costs weakly less than the income endowment.
3. So if $\mathbf{p}^*$ is an exact c.e., each student must be able to afford some bundle in any $N+1$-way split.
4. Hence, each student must be able to afford some bundle weakly preferred to her $N+1$-maximin share.

Argument is a bit messier because $\mathbf{p}^*$ might be an approximate c.e. The proof exploits the Kakutani fixed-point step from the proof of Theorem 1.

Theorem 3: Approximate CEEI Guarantees that Envy is Bounded by a Single Good

We know that exactly equal incomes guarantees exact envy-freeness, because all students have the same choice set.

Theorem 3: if $\beta < \frac{1}{k-1}$ then $\mathbf{x}^*$ satisfies envy bounded by a single good

- Intuition: suppose s_i envies s_j. Then

$$1 \leq b_i^* < \mathbf{p}^* \cdot x_j^* \leq b_j^* \leq \frac{k}{k-1}$$

- Since x_j^* contains at most k goods, one of them must cost at least $\frac{1}{k-1}$. s_i can afford the bundle formed by removing this good from x_j^*
- By revealed preference, s_i must weakly prefer her own bundle to the bundle formed by removing this single good from x_j^*, so her envy is bounded.

Notice that budget inequality plays slightly different roles in the two proofs.

The Approximate CEEI Mechanism

1. Agents report their preferences
2. Agents are given **approximately** equal budgets of an artificial currency (uniform draws from $[1, 1+\beta]$ for β suitably small)
3. We find an item price vector $\mathbf{p}^*$ such that, when each agent i is allocated his favorite bundle in his budget set $\{x \in \Psi_i : \mathbf{p}^* \cdot x \leq b_i^*\}$ the market **approximately** clears (market-clearing error as small as possible, and certainly no larger than $\frac{\sqrt{\sigma M}}{2}$)
4. We allocate each agent their demand at $\mathbf{p}^*$

Note 1: choosing prices uniform randomly ensures that the procedure is strategyproof in a large market. There are other possible tie-breaking rules that preserve incentives in this way.

Note 2: it is possible to add a step in which we first seek an exact CEEI.

Strategyproof in the Large

- My procedure is not strategyproof
- But, any student who regards prices as exogenous should report their preferences truthfully
  - Formally, the mechanism is strategyproof in an appropriately-defined continuum economy (Theorem 4).
  - Note: can't execute Pareto-improving trades ex-post without undermining SPITL
- By contrast, consider the HBS mechanism, or the Boston mechanism for school choice
- Even if we hold fixed the analog of prices in these mechanisms, agents still should not report their preferences truthfully.
- All course-allocation mechanisms currently found in practice are manipulable even in continuum markets.

Which Market Designs are Strategyproof in the Large?

Manipulable in Large Markets	**SP in Large Markets**
Bidding Points Mechanism	Assignment Exchange
Boston Mechanism	Deferred Acceptance
Generalized Second Price	Double Auctions
HBS Mechanism	Probabilistic Serial
Discriminatory Auctions	Uniform Price Auctions

- All course-allocation mechanisms currently found in practice are manipulable even in large markets
  - HBS: empirical evidence that this matters for welfare
- By contrast, many widely used non-strategyproof mechanisms are strategyproof in the large
  - Of course, in several instances we have a much more highly detailed understanding of incentives away from the limit (e.g., deferred acceptance, double auctions)

Properties of the Approximate CEEI Mechanism

Efficiency

- Ex-post efficient with respect to the allocated goods.

Fairness

- Symmetric

- N+1 Maximin Share Guaranteed

- Envy Bounded by a Single Good

Incentives

- Strategyproof in a Large Market

Relationship to Random Serial Dictatorship

Single-Unit Demand

- The Approximate CEEI Mechanism coincides with Random Serial Dictatorship
- Both satisfy maximin-share guarantee and envy bounded by a single good
- Dictatorships frequently used in practice (school choice, housing assignment)

Multi-Unit Demand

- The mechanisms are importantly different.
- Suppose students require at most k objects. RSD corresponds to an exact competitive equilibrium ($\alpha = 0$) from budgets of

$$\mathbf{b}^{RSD} = (1, k+1, (k+1)^2, (k+1)^3, ..., (k+1)^{N-1})$$

- Dictatorships not observed in practice

Table 2: Comparison of Alternative Mechanisms

Mechanism	Efficiency (Truthful Play)	Outcome Fairness (Truthful Play)	Procedural Fairness	Incentives	Preference Language
Approximate CEEI Mechanism	Pareto Efficient w/r/t Allocated Goods Allocation error is small for practice and goes to zero in the limit	N+1 – Maximin Share Guaranteed Envy Bounded by a Single Good	Symmetric	Strategyproof in the Large	Ordinal over Schedules
Random Serial Dictatorship (Sec 7.1)	Pareto Efficient	Worst Case: Get k worst Objects	Symmetric	Strategyproof	Ordinal over Schedules
Multi-unit generalization of Hylland Zeckhauser Mechanism (Sec 7.2)	If vNM preferences are described by assignment messages, ex-ante Pareto efficient	If preferences are additive separable, envy bounded by the value of two goods Worst Case: Get Zero Objects	Symmetric	If vNM preferences are described by assignment messages, Strategyproof in the Large	Assignment messages
Bidding Points Mechanism (Sec 7.3)	If preferences are additive-separable, Pareto Efficient but for quota issues described in Unver and Sonmez (forth.)	Worst Case: Get Zero Objects	Symmetric	Manipulable in the Large	Cardinal over Items
Unver-Sonmez (forth.) Enhancement to Bidding Points Mechanism	If preferences are additive-separable, Pareto Efficient	Worst Case: Get Zero Objects	Symmetric	Bidding Phase: Manipulable in the Large Allocation Phase: Strategyproof in the Large	Bidding Phase: Cardinal over Items Allocation Phase: Ordinal over Items
HBS Draft Mechanism (Sec 8.2)	If preferences are responsive, Pareto Efficient with respect to the reported information (i.e., Pareto Possible)	If preferences are responsive and k=2, Maximin Share Guaranteed If preferences are responsive, Envy Bounded by a Single Good	Symmetric	Manipulable in the Large	Ordinal over Items
Bezakova and Dani (2005) Maximin Utility Algorithm	If preferences are additive-separable, ideal fractional allocation is Pareto efficient. Realized integer allocation is close to the fractional ideal.	Worst Case: Get approximately zero objects (if a hedonist and all other agents are depressives)	Symmetric	Manipulable in the Large	Cardinal over items
Brams and Taylor (1996) Adjusted Winner	If preferences are additive-separable, Pareto Efficient	Worst Case: Get Zero Objects	Symmetric	Manipulable in the Large	Cardinal over Items
Herreiner and Puppe (2002) Descending Demand Procedure	Pareto Efficient	Does not satisfy Maximin Share Guarantee or Envy Bounded by a Single Object	Symmetric	Manipulable in the Large	Ordinal over Schedules
Lipton et al (2004) Fair Allocation Mechanism	Algorithm ignores efficiency	If preferences are additive separable, Envy Bounded by a Single Good	Symmetric	Manipulable in the Large	Cardinal over items
UChicago Primal-Dual Linear Programming Mechanism (Graves et al 1993)	Pareto Efficient when preference-reporting limits don't bind	Worst Case: Get Zero Objects	Symmetric	Manipulable in the Large	Cardinal over a Limited Number of Schedules

Ex-Ante Welfare Performance of Approximate CEEI

Overview

- The Approximate CEEI Mechanism has an element of randomness: the budgets.
- Efficiency ideally should be assessed ex-ante, not ex-post
- A follow-on paper (with Che, Kojima and Milgrom) develops a course-allocation procedure that is ex-ante efficient when students' vNM preferences for courses can be described by assignment messages (Milgrom, 2009). Some tradeoffs:
  - no longer satisfy the outcome fairness criteria;
  - less flexibility in permissible schedule sets;
  - need to assume risk neutrality
- In this paper, I assess ex-ante efficiency empirically in a specific course-allocation environment

Computational Analysis - Algorithm

Theorem 1 is non constructive, and implementing the Approximate CEEI Mechanism is non-trivial. There are two key challenges:

1. Calculating excess demand at a particular price ($z(\mathbf{p})$) is NP Hard – each agent must solve a set-packing problem
2. Price space is large. So even if $z(\mathbf{p})$ were easy to compute, finding an approximate zero is a difficult search problem

Othman, Budish and Sandholm (2009) develop a computational procedure that overcomes these challenges in life-size problems.

1. Demands are calculated using an integer program solver, CPLEX
2. We use a method called "Tabu Search" to find an approximate zero. Departure point is the Tatonnement process $\mathbf{p}^{t+1} = \mathbf{p}^t + z(\mathbf{p}^t)$

The algorithm can currently handle "semester-sized" economies in which students consume 5 courses. Each run takes 1 hour.

Computational Analysis - Data and Key Assumptions

- HBS data: preferences are ordinal over individual courses.
- To convert into utilities from bundles I make two substantive assumptions
  - A1: Additive-Separable Preferences (no comps or subs)
  - A2: Students care about the "Average Rank" of the courses they receive (e.g. 2nd + 3rd favorite better than 1st + 5th favorite)
  - Theory can handle more complex preferences but A1 and A2 seem reasonable given data incompleteness
- A3: Students report their preferences truthfully under Approximate CEEI
  - Caveat: no way to validate whether 916 students is "large"

Ex-Ante Welfare Performance of Approximate CEEI

Summary of Findings

Finding #1: Market-clearing error is small

- Max observed error in a semester is $\sqrt{14}$, and the mean is $\sqrt{6}$ (± 1 seat in each of 6 courses), versus 4500 total seats allocated.
- Implication: ex-post inefficiency is small

Finding #2: Individual students' outcomes seem not to vary much with the random budgets

- 50% of students: no variation in utility over the random budget draws. Max observed variation is 5 ranks
- Implication: ex-post efficiency is a reasonable proxy for ex-ante efficiency (unlike for RSD)

Finding #3: Distribution of utilitites f.o.s.d.'s that from HBS's own mechanism, and s.o.s.d.'s that from RSD.

- Implication: a utilitarian social planner should prefer Approximate CEEI to either of these alternatives

Isn't CEEI Already Used in Practice?

A widely used course-allocation procedure is the Bidding Points Mechanism (see Sonmez and Unver, forth):

1. Each student is given an equal budget of artificial currency, say 1000 points.
2. Students express preferences by bidding for individual classes, the sum of their bids not to exceed 1000
3. For a course with q seats, the q highest bidders get a seat (modulo some quota issues)

- Schools describe the q^{th} highest bid as the "price", and the procedure as a "market".
- To the casual observer, this procedure looks like CEEI ... which we know need not exist!

The Bidding Points Mechanism is not a CEEI

- Two mistakes: wrong prices, wrong demands
- Conceptual error: the market treats fake money as if it were real money that enters the utility function.
- Correct "fake money" demand

$$x_i^* = \arg\max_{x \in \Psi} (u_i(x) : \mathbf{p}^* \cdot x \leq b_i)$$

- Incorrect "real money" demand

$$x_i^* = \arg\max_{x \in \Psi} (u_i(x) - \mathbf{p}^* \cdot x)$$

- Some virtues: always exist, easy to compute ...

What Goes Wrong in the BPM: Incentives

Incentives to misreport are easy to see.

- Three courses, $\{A, B, C\}$
- Suppose $u_{Alice} = (700, 200, 100)$ and $\mathbf{p}^* = (800, 300, 150)$
- Bid truthfully $\longrightarrow$ get zero courses
- BR: bid $\widehat{u}_{Alice} = (801, 0, 151)$

What's so bad about this?

- Alice simply tricked a "real money" demand function into behaving like a "fake money" demand

What Goes Wrong in the BPM: Fairness

Answer: Betty!

- Alice's bid of 801 of A displaces some student who bid 800
- This student now wastes 800 of her points; at best, gets correct demand given a budget of 200.

Proposition 10: Suppose a CEEI actually exists

- Truthful play $\not\Rightarrow$ CEEI
- Eqm play $\not\Rightarrow$ CEEI

Proposition 11:

- Truthful play $\Rightarrow$ Some students get ex-post utility of zero
- Eqm play $\Rightarrow$ Some students get ex-post utility of zero

By contrast: A-CEEI yields an exact CEEI whenever one exists, and fairness theorems prevent highly unfair outcomes.

Summary of CAP: A-CEEI

- Dictatorship theorems: there is no perfect mechanism for combinatorial assignment. Compromise is needed.
- I propose criteria that constitute an attractive compromise of competing objectives
  - Outcome fairness: Maximin Share Guarantee, Envy Bounded by a Single Good
  - Incentives: Strategyproof in a Large Market
- I construct a specific mechanism that satisfies the criteria while maintaining approximate ex-post efficiency
  - Adapt the CEEI to an indivisible-goods environment
  - Existence requires approximating both "CE" and "EI"
- Ex-ante efficiency performance compelling on data

Conclusion

- Theory on CAP at a dead end; mostly negative results
- First, look to practice. HBS mechanism yields several lessons.
- Then, develop a new theoretical solution.
- Rather than discrete goods / game theory, inspired by divisible goods / general equilibrium theory
- Many outstanding questions, both theoretical and practical
- Grant to develop software.
- If history is any guide, implementation in practice will yield new directions for theory.